Some Formulas

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$$f(y\mid\theta_i,\varphi) = \exp\left\{[y\theta_i - b(\theta_i)]\Big/\left(\frac{\varphi}{w_i}\right) + c_i(y,\varphi)\right\}$$

(a) $b'(\theta_i) = E(Y_i) \equiv \mu_i$

(b) $\left(\frac{\varphi}{w_i}\right) b''(\theta_i) = \operatorname{var}(Y_i)$

$$g(\mu_i) \equiv \eta_i = \begin{cases} \underset{\sim}{x}_i^t \underset{\sim}{\beta} \\ \underset{\sim}{x}_i^t \underset{\sim}{\beta} + c_i \leftarrow \text{offset (Poisson)} \end{cases}$$

canonical link : $\eta_i = \theta_i$

$$L = \text{log-lik} = \frac{1}{\varphi}\sum_{i=1}^n w_i[y_i\theta_i - b(\theta_i)] + \sum_{i=1}^n c(y_i,\varphi)$$

$\Rightarrow$

$$\frac{\partial L}{\partial\beta_j} = \frac{1}{\varphi}\sum_{i=1}^n w_i[y_i - b'(\theta_i)]\cdot\frac{\partial\theta_i}{\partial\beta_j}$$

$$\frac{\partial^2 L}{\partial\beta_k\partial\beta_j} = -\frac{1}{\varphi}\sum_{i=1}^n w_i\left\{b''(\theta_i)\left(\frac{\partial\theta_i}{\partial\beta_k}\right)\left(\frac{\partial\theta_i}{\partial\beta_j}\right) - [y_i - b'(\theta_i)]\frac{\partial^2\theta_i}{\partial\beta_k\partial\beta_j}\right\}$$

Using $\frac{\partial\theta_i}{\partial\beta_j} = \frac{d\theta_i}{d\mu_i}\cdot\frac{d\mu_i}{d\eta_i}\cdot\frac{\partial\eta_i}{\partial\beta_j} = \frac{1}{b''(\theta_i)}\cdot\frac{1}{g'(\mu_i)}\cdot x_{ij}$ with (a),(b):

$$\frac{\partial L}{\partial\beta_j} = \sum_{i=1}^n (y_i - \mu_i)\cdot\frac{1}{g'(\mu_i)\cdot\operatorname{var}(Y_i)}\cdot x_{ij}$$

$$\frac{\partial^2 L}{\partial\beta_k\partial\beta_j} = -\sum_{i=1}^n \frac{x_{ij}\,x_{ik}}{\operatorname{var}(Y_i)\cdot[g'(\mu_i)]^2} + \underbrace{\frac{1}{\varphi}\sum_{i=1}^n w_i(y_i-\mu_i)\frac{\partial^2\theta_i}{\partial\beta_k\partial\beta_j}}_{\text{II}}$$

Fisher scoring : $\underset{\sim}{\beta}_{(m+1)} = \underset{\sim}{\beta}_{(m)} + (X^t A_m^{-1} X)^{-1} X^t A_m^{-1} \underset{\sim}{r}_m$ where

$A = \operatorname{diag}[g'(\mu_i)^2\cdot\operatorname{var}(Y_i)]$, $\underset{\sim}{r} = (g'(\mu_i)\cdot(y_i - \mu_i))$

Asymptotic Distribution:

$(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta}) \approx M\cdot(\underset{\sim}{Y} - \underset{\sim}{\mu})$ for $M = (X^t A^{-1} X)^{-1} X^t A^{-1} G$, $G = \operatorname{diag}(g'(\mu_i))$

$\Rightarrow$ $\hat{\underset{\sim}{\beta}} \approx N(\underset{\sim}{\beta}, W)$ for $W = \left\{X^t[\operatorname{diag}(g'(\mu_i)^2\cdot\operatorname{var}(Y_i))]^{-1}X\right\}^{-1}$