Proof that $E(Y) = b'(\theta)$, $var(Y) = \frac{\varphi}{w} b''(\theta)$ for exp. family (11.1)

$$f(y|\theta,\varphi) = e^{[y\theta - b(\theta)]/(\frac{\varphi}{w}) + c(y,\varphi)}$$

① $$0 = \frac{\partial}{\partial\theta}\int f(y|\theta)\, dy = \int \frac{\partial}{\partial\theta} f(y|\theta)\, dy$$

$$= \int \frac{\partial}{\partial\theta}\left\{\frac{[y\theta - b(\theta)]}{(\varphi/w)} + c(y,\varphi)\right\} f(y)\, dy \quad (**)$$

$$= \int \left[\frac{y - b'(\theta)}{(\varphi/w)} + 0\right] f(y)\, dy = \underbrace{\frac{w}{\varphi}\int [y - b'(\theta)]\, f(y)\, dy}_{(*)}$$

$$\Rightarrow E(Y) = b'(\theta).$$

② $$0 = \frac{\partial^2}{\partial\theta^2}\int f(y|\theta)\, dy \overset{(*)}{=} \frac{\partial}{\partial\theta}\frac{w}{\varphi}\int \overset{f}{[y - b'(\theta)]}\, \overset{g}{f(y|\theta,\varphi)}\, dy$$

$$\Rightarrow 0 = \int \frac{\partial}{\partial\theta}\overset{f'}{[y - b'(\theta)]}\cdot \overset{g}{f(y|\theta,\varphi)}\, dy + \int [y - b'(\theta)]\, \underbrace{\frac{\partial}{\partial\theta} f(y|\theta,\varphi)}_{(**)}\, dy$$

$$= \int -b''(\theta)\, f(y|\theta)\, dy + \int (y-\mu)\frac{(y-\mu)}{\varphi/w}\cdot f(y|\theta,\varphi)\, dy$$

$$= -b''(\theta) + \frac{w}{\varphi} var(Y)$$

$$\Rightarrow var(Y) = \frac{\varphi}{w} b''(\theta)$$

Proof that for the exponential family,

$$E(Y) = b'(\theta)$$

$$var(Y) = \frac{\varphi}{w} b''(\theta).$$